Lecture 10: Antiderivatives

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A function F is called an **antiderivative** of f on an interval I if F'(x)=f(x) for all x in the interval I.
To put it simply: we're doing the *reverse* of differentiation.

Example:
$f(x) = x^2 = F'(x)$
When we differentiate using the power rule, we subtract 1 from the exponent. So when we *anti*differentiate, we must add one to the exponent.
$= x^{2+1} = x^3$
AND when we differentiate, we multiply by the derivative in the exponent: in this case, the exponent is 3. So when we *anti*differentiate, we divide by the derivative of the exponent, like this:
$= \frac{x^3}{3}$
We've discovered that the anti-derivative of $F'(x) = x^2$ is $F(x) = \frac{x^3}{3}$.
BUT, F(x) can also technically be $\frac{x^3}{3} + 7$ or $\frac{x^3}{3} + 19$. So, we need to add a constant value C to our expression.
$F(x) = \frac{x^3}{3} + C$, where C is any constant

Theorem If F is an antiderivative of f on I, then every function F(x)+C is also an antiderivative of f on I.

Example:

1. $f(x) = \sin(x)$
   Guess: $F(x) = -\cos(x) + \mathrm{C}$
   Test: $F'(x) = -(-\sin(x)) + \mathrm{C} = \sin(x)$ ✓

2. $f(x) = \frac{1}{x}$
   $F(x) = \ln(x) + \mathrm{C}$
   BUT $f(x)$ is defined on all real numbers except for 0 and $\ln(x)$ is only defined for positive x values.
   So:
   if $x < 0$: $(\ln(-x))' = \frac{1}{-x} * \underset{\text{derivative of } -x}{(-1)} = \frac{1}{x}$

   New rule: $(\ln(|x|))' = \frac{1}{x}$
   So the antiderivative of $\frac{1}{x}$ is $\ln(|x|) + \mathrm{C}$.
   don't forget the absolute value!

Table of antiderivatives: (no +C)

function	particular antiderivative
$c * f(x)$	$c * F(x)$
$f(x) + g(x)$	$F(x) + G(x)$
x^n	$\frac{x^{n+1}}{n+1}$

$\frac{1}{x}$	$\ln(\lvert x\rvert)$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x) = \frac{1}{\cos^2(x)}$	$\tan(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x) = \arcsin(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$
$\sec(x) * \tan(x)$	$\sec(x)$

<u>Examples</u>:

1. $f(x) = e^{3x}$

   $F(x) = \frac{e^{3x}}{3} + C$

   (test: $F'(x) = \frac{1}{3}(e^{3x} * 3) = e^{3x}$)
2. $f(x) = 2\cos(7x)$

   $F(x) = \frac{2\sin(7x)}{7} + \mathrm{C}$
3. Find all antiderivatives of:

   $g(x) = 3\cos(x) + \frac{\sqrt{x} - 3x^5}{x}$

   Approach: break up sum; find antiderivative of $3\cos(x)$ first:

   $G(x) = 3\sin(x) + \_\_\_\_\_\_$

   Rewrite $\frac{\sqrt{x}-3x^5}{x}$ as $\frac{\sqrt{x}}{x} - \frac{3x^5}{x} = x^{-\frac{1}{2}} - 3x^4$

   $G(x) = 3\sin(x) + 2x^{\frac{1}{2}} - \frac{3}{5}x^5 + \mathrm{C}$

   (Just add C at the end instead of adding C_1, C_2, C_3, etc. in every step ($C=C_1+C_2+C_3$)

4. $g(x) = e^x + 12x^2 + \frac{1}{1+x^2}$

   from table

   $G(x) = e^x + 4x^3 + \arctan(x) + \mathrm{C}$

5. Antiderivative of $f(x) = \frac{3x^2+2}{x^3+2x}$

   First: spot that $3x^2 + 2$ is the derivative of $x^3 + 2x$!

   So we try:

   $F(x) = \ln(\lvert x^3 + 2x\rvert) + \mathrm{C}$

   test:

   $F'(x) = \frac{1}{x^3+2x} * (3x^2+2) + 0$

   $= \frac{3x^2+2}{x^3+2x}$

<u>Application to physics</u>

$v'(t) = a(t)$ $\quad$ $s'(t) = v(t)$

velocity $\quad$ acceleration $\quad$ location function

$s''(t) = a(t)$

Example:
Given $a(t) = 6t + 4$, find $v(t)$ at time $t = 3s$.
$v'(t) = a(t)$, so $v(t)$ is an antiderivative of a(t).
$v(t) = A(t) = 3t^2 + 4t + \mathrm{C}$

In order to determine C, we need more information. Usually we're given the initial value of C. For example, say $v(0) = 6\frac{m}{s}$
Now with this information, we can specify a particular antiderivative (compute a value for C)

$$v(0) = 3(0)^2 + 4(0) + \mathrm{C} = \mathrm{C} = 6\frac{\mathrm{m}}{\mathrm{s}}$$

So, $v(t) = 3t^2 + 4t + 6$ is a particular antiderivative.
Now, $v(3) = 3(3)^2 + 4(3) + 6 = 27 + 12 + 6 + 45\frac{m}{s}$

Logarithmic Differentiation

(New technique for differentiating)

ex. Differentiate $f(x) = \frac{x^{\frac{3}{4}}(\sqrt{x^2+1})}{(3x+2)^5}$

Very long solution if we use quotient rule

Instead, take the ln of both sides:

$$\ln f(x) = \ln\left(\frac{x^{\frac{3}{4}}(\sqrt{x^2+1})}{(3x+2)^5}\right)$$

$$= \ln(\frac{x^3}{4}) + \ln\left(\sqrt{x^2+1}\right) - \ln((3x+2)^5)$$

...

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

now use implicit differentiation:

$$\frac{1}{f(x)} * f'(x) = \frac{3}{4x} + \frac{1}{2} * \frac{1}{x^2+1}(2x) - \frac{5}{3x+2}(3)$$

$$f'(x) = f(x)\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

replace with function definition

$$f'(x) = \frac{x^{\frac{3}{4}}(\sqrt{x^2+1})}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

We could have used the quotient rule, but sometimes this is the only way that works.

Example

$y(x) = x^{\sqrt{x}}$
$\ln(y(x)) = \ln(x^{\sqrt{x}})$
$\ln(y(x)) = \sqrt{x}\ln(x)$ ← product rule

implicit differentiation

$$\frac{1}{y(x)}y'(x) = \frac{1}{2\sqrt{x}}\ln(x) + \sqrt{x}\left(\frac{1}{x}\right)$$

express $y'(x) = y(x)\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$

$$= x^{\sqrt{x}}\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$